

[Methods for solving total differential equation in three variables.]

• Method (2) Solution by inspection.

When the condition of integrability is satisfied, it may be possible that by rearranging the terms, it becomes exact and the solution is found readily.

Q. (1) Solve:  $(yz+zx)dx + (zx-yz)dy + (xy-yx)dz = 0$

Solution:— By comparing the given equation to the total diff. eqn.  $Pdx + Qdy + Rdz = 0$ , we find that

$$P = yz+zx, Q = zx-yz, R = xy-yx$$

$$\therefore P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$$

$$= (yz+zx)(x-2-z+2) + (zx-yz)(y-z) + (xy-yx)(z-2) = 0$$

i.e. the condition of integrability is satisfied.

Hence, the eqn is exact and can be written as

$$yzdx + zx dy + xy dz - yz dy + xy dz - yz dz = 0$$

$$\Rightarrow (yzdx + zx dy + xy dz) + 2xdx - 2(yz dy + yz dz) = 0$$

Integrating,  $\boxed{xyz + x^2 - 2yz = C}$

which is the required solution

Q. (2) Solve  $(y+z)dx + dy + dz = 0$

Here,  $P = y+z, Q = 1, R = 1$

$$\text{then } P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$$

$$= (y+z)(0-0) + 1(0-1) + 1(1-0) = 0$$

Hence the condition of integrability is satisfied.

The given eqn can be written as

$$dx + \frac{dy+dz}{y+z} = 0$$

Integrating,

$$x + \log(y+z) = \log c \Rightarrow \log e^x + \log(y+z) = \log c$$

$$\Rightarrow \log\{e^x(y+z)\} = \log c$$

$$\Rightarrow \boxed{y+z = c e^x} - \text{which is required solution.}$$

Q. ③ Solve  $(yz \log z)dx - (zx \log z)dy + xy dz = 0$

Here  $P = yz \log z$ ,  $Q = -zx \log z$ ,  $R = xy$

$$\text{Then, } P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$$

$$= yz \log z (-x \log z - x - x) - zx \log z (y - y \log z - y) \\ + xy (z \log z + z \log z) = 0$$

$\therefore$  The condition of integrability is satisfied.

Now, dividing the given equation by  $xyz \log z$ ,

we get  $\frac{dx}{x} - \frac{dy}{y} + \frac{dz}{\log z} = 0$

Integrating, we get

$$\log x - \log y + \log(\log z) = \log c$$

$$\text{or, } \log\left(\frac{x}{y} \cdot \log z\right) = \log c$$

$$\therefore \boxed{x \log z = cy}$$

which is required solution.

Q. (4) Verify that the equation

$$(y-z)(y+z-2x)dx + (z-x)(z+x-2y)dy + (x-y)(x+y-2z)dz = 0$$

is exact and find the solution.

Solution: — Comparing the given equation to the standard equation  $Pdx + Qdy + Rdz = 0$ ,

$$\text{we get } P = (y-z)(y+z-2x), \quad Q = (z-x)(z+x-2y)$$

$$\text{and } R = (x-y)(x+y-2z).$$

$$\text{Now, } \frac{\partial P}{\partial y} = 2y - 2x, \quad \frac{\partial Q}{\partial x} = 2y - 2x, \quad \frac{\partial R}{\partial z} = 2z - 2y$$

$$\frac{\partial R}{\partial y} = 2z - 2y, \quad \frac{\partial Q}{\partial z} = 2x - 2z, \quad \frac{\partial P}{\partial z} = 2x - 2z$$

In this way, we see that

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \quad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y} \text{ and } \frac{\partial R}{\partial x} = \frac{\partial P}{\partial z}$$

Hence, the given equation is exact.

Now, the given equation can be written as

$$(y^2 + yz - 2xy - yz - z^2 + 2xz)dx + (z^2 + xy - 2yz - xz - x^2 + 2xy)dy + (x^2 + xy - 2xz - xy - y^2 + 2yz)dz = 0$$

$$\Rightarrow (y^2 dx + 2xy dy) - (z^2 dx + 2xz dz) + (z^2 dy + 2zy dz) - (x^2 dy + 2xy dx) + (x^2 dz + 2xz dx) - (y^2 dz + 2yz dy) = 0$$

Integrating,

$$y^2 x - z^2 x + z^2 y - x^2 y + x^2 z - y^2 z = C$$

$$\Rightarrow \boxed{x(y^2 - z^2) + y(z^2 - x^2) + z(x^2 - y^2) = C}.$$

which is the required solution.

Q.(5) Solve  $x dx + y dy - \sqrt{a^2 - x^2 - y^2} dz = 0$  48

Condition for integrability can be shown readily and easily.

Now, the equation can be written as

$$\frac{x dx + y dy}{\sqrt{a^2 - x^2 - y^2}} = dz$$

$$\Rightarrow \frac{x dx + y dy}{\sqrt{a^2 - (x^2 + y^2)}} = dz$$

$$\text{i.e. } \frac{d(x^2 + y^2)}{\sqrt{a^2 - (x^2 + y^2)}} = 2 dz$$

Integrating,

$$\sin^{-1}\left(\frac{x^2 + y^2}{a}\right) = 2z + C$$

$$\text{or, } \boxed{x^2 + y^2 = a \sin(2z + C)}$$

which is the required solution.

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